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Star Product and *q*-Deformation of Grassmann and Symmetric Algebras

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We review the relation between the star product on a Poisson–Lie group and the quantum Yang–Baxter equation. We define the q-deformed Grassmann and q-deformed symmetric algebras on a vector space V, and prove that a star product on a triangular (simple quasitriangular) Poisson–Lie group G determines a q-deformation of both the symmetric and Grassmann algebras over a dual of a g-module where g is the corresponding triangular (simple quasitriangular) Lie bialgebra of the Poisson–Lie group G.

1. INTRODUCTION

In the last two decades there has been great interest in studying quantum groups. These structures emerge in several contexts, especially in statistical physics (Faddeev, 1984), factorizable S-matrix theory (Baxter, 1982), conformal field theory (Alvarez-Gaume, 1990) and Chern–Simons theory (Kac and Raina, 1987). They first appeared as quantum algebras, i.e., as one-parameter deformations of the universal enveloping algebras of complex simple Lie algebras. Then it was realized (Drinfeld, 1987) that, mathematically, quantum groups are Hopf algebras, more precisely quantum groups can be seen as noncommutative generalizations of topological spaces which have a group structure. Such a structure induces an Abelian Hopf algebra (Abe, 1980) structure on the algebra of smooth functions defined on the group. Quantum groups are defined then as non-Abelian Hopf algebras (Takhtajan, 1989). They can be generated by deforming the Abelian product of the Hopf algebra of smooth functions into a non-Abelian one (*-product), using the so-called quantization deformation (Bayen *et al.*, 1978) or star-deformation procedure.

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This approach was used by Moreno and Valero (1992) to prove same theorems of Drinfeld about solutions of the triangular quantum Yang–Baxter equation and corresponding quantum groups, and by Mansour (1997) to show that a star product on a Poisson–Lie group leads to an h-deformation of the corresponding Lie algebra.

In the present work we generalize Akhoumach and Mansour (1996) and show that a star product on a triangular (simple quasitriangular) Poisson–Lie group G [the Poisson–Lie structure is given by an r-matrix which satisfies the classical (generalized) Yang–Baxter equation] determines a q-deformation of both symmetric and Grassmann algebras over a dual of a g-module V, where g is the corresponding Lie algebra of the Lie group G.

2. THE STAR PRODUCT AND QUANTUM YANG–BAXTER EQUATION

Let G be a (simple) Lie group, $\mathbf{g}$ its (simple) Lie algebra, and let $r \in \Lambda^2(\mathbf{g})$ satisfy the generalized Yang–Baxter equation (GYBE), i.e.,

$$[r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = -[t_{13}, t_{23}] \tag{2.1}$$

where t is a symmetric and ad-invariant element of $\mathbf{g} \otimes \mathbf{g}$, i.e.,

$$t_{21} = t_{12}, \qquad [\Delta(x), t] = 0 \quad \text{for all} \quad x \in \mathbf{g} \tag{2.2}$$

where $\Delta(x) = x \otimes 1 + 1 \otimes x$.

If $t = 0$, we say that $r \in \Lambda^2(g)$ satisfies the classical YBE,

$$[r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = 0$$

We recall that the Poisson bracket on the associative algebra $C^\infty(G)$ of C^∞ functions on G with respect to the usual product associated to the r-matrix, $r = r^{\mu\nu} X_\mu \otimes X_\nu / r^{\mu\nu} = -r^{\nu\mu}$, is given by

$$\{\phi, \psi\}(g) = r^{\mu\nu}(X^l_{\mu/g}(\phi) \cdot X^l_{\nu/g}(\psi) - X^r_{\mu/g}(\phi) \cdot X^r_{\nu/g}(\psi))$$

where $\{X^l_\mu$ (resp., $X^r_\mu)\}$ is the basis of left (resp., right) invariant vector fields defined from $\{X_\mu\}$ (basis of $\mathbf{g}$) by left (resp., right) translation.

We recall also according to Moreno and Valero (1992) that a star product on a Poisson–Lie group G is defined as a bilinear map

$$\begin{aligned} C^\infty(G) \times C^\infty(G) &\overset{*}{\to} C^\infty(G)[[h]] \\ (\varphi, \psi) \to \varphi * \psi &= \sum_{k \geq 0} C_k(\varphi, \psi)\, h^k \end{aligned} \tag{2.3}$$

where

$$C_0(\varphi, \psi) = \varphi \cdot \psi$$

$$C_1(\varphi, \psi) = \{\varphi, \psi\}$$

and the elements C_k are bidifferential operators on $C^\infty(G)$ null on the constants such that

$$\varphi * 1 = 1 * \varphi = 1 \tag{2.4}$$

$$(\varphi * \psi) * \phi = \varphi * (\psi * \phi) \tag{2.5}$$

$$\Delta(\varphi * \psi) = \Delta(\varphi) * \Delta(\psi) \tag{2.6}$$

where Δ is the usual coproduct on $C^\infty(G)$ defined by

$$\Delta(\varphi)(x, y) = \varphi(x \cdot y) \tag{2.7}$$

Takhtajan (1989) gives the desired star product by the following expression:

$$\varphi * \psi = \mu((F^{-1})^r(F^l)(\varphi \otimes \psi)) \tag{2.8}$$

where

$$F = 1 + \frac{h}{2} r + \sum_{k \geq 2} F_k h^k \in U(\mathbf{g})^{\otimes 2}[[h]] \tag{2.9}$$

[$F^l(F^r)$ is the left (right) invariant bidifferential operator defined from F by left (right) translation] satisfies the following equation:

$$(\Delta_0 \otimes id)F \cdot (F \otimes 1) = \Phi \cdot (id \otimes \Delta_0)F \cdot (1 \otimes F) \tag{2.10}$$

[Δ_0 is the usual coproduct of the enveloping algebra $U(\mathbf{g})$], where Φ is Ad_G-invariant, i.e.,

$$[X \otimes 1 \otimes 1 + 1 \otimes 1 \otimes X + 1 \otimes X \otimes 1, \Phi] \qquad \text{for any} \quad X \in \mathbf{g}$$

This permits a quasitriangular quasi-Hopf algebra (quasi-quantum group) (Drinfeld, 1990; Mansour, 1998) structure on the enveloping algebra $(U(\mathbf{g})[[h]], \Delta_0, \Phi, R = e^{ht/2})$, where t is the symmetric element (2.2).

Drinfeld (1990), shows that the quasitriangular structure on the quantized enveloping algebra of a simple Lie algebra $(U(\mathbf{g})[[h]], \Delta_F, \varepsilon, S_F)$ is given by the R-matrix,

$$R_F = F_{21}^{-1} e^{ht/2} F_{12}$$

which satisfies the quasitriangular quantum Yang–Baxter equation (QYBE)

$$R_F(x, y) R_F(x, z) R_F(y, z) = R_F(y, z) R_F(x, z) R_F(x, y) \tag{2.11}$$

If $t = 0$ (the r-matrix is a solution of the classical Yang–Baxter equation), then we find the R-matrix given in Drinfeld (1983) and Moreno and Valero (1992) by

$$S(x, y) = F^{-1}(y, x)F(x, y)$$

which satisfies the triangular QYBE

$$S(x, y)S(x, z)S(y, z) = S(y, z)S(x, z)S(x, y)$$
$$S(x, y)S(y, x) = 1$$

3. DEFORMATION OF BOTH GRASSMANN AND SYMMETRIC ALGEBRAS

3.1. Deformation of the Symmetric Algebra

Let V be a vector space over the field $\mathbb{C}$ such that $\dim_{\mathbb{C}} V = n$, and let $(e_1, e_2, \ldots, e_n)$ be an arbitrary basis of V. Let then $(\sigma^1, \sigma^2, \ldots, \sigma^n)$ be the basis of the dual space V^* such that

$$\sigma^i(e_j) = \delta^i_j \tag{3.1}$$

Recall that if σ^i $(i = 1, \ldots, n)$ are linear forms on V, then their symmetric product is as usual given by

$$\sigma^i \vee \sigma^j = \sigma^i \otimes \sigma^j + \sigma^j \otimes \sigma^i \tag{3.2}$$

We define the symmetric q-product of two linear forms σ^i and σ^j, written $\tilde{\vee}$, by

$$\sigma^i \tilde{\vee} \sigma^j = \sigma^i \otimes \sigma^j + \Lambda^{ij}_{kl}\sigma^k \otimes \sigma^l \tag{3.3}$$

where the elements Λ^{ij}_{kl} are entries of some matrix $\Lambda \in \mathrm{End}_{\mathbb{C}}(V^* \otimes V^*)$:

$$\Lambda(\sigma^i \otimes \sigma^j) = \Lambda^{ij}_{kl}\sigma^k \otimes \sigma^l \tag{3.4}$$

we demand that all Λ^{ij}_{kl} be scalars depending on an arbitrary complex number q referred to as the deformation parameter, such that in the limit $q = 1$ we have

$$\Lambda^{ij}_{kl}(q = 1) = \delta^i_l\delta^j_k \tag{3.5}$$

A general q-two-tensor reads

$$t^{(2),q} = t_{ij}\sigma^i \tilde{\vee} \sigma^j \tag{3.6}$$

where t_{ij} are complex numbers.

To construct a q-tensor of higher order, we introduce the overlap between the two operations $\otimes$ and $\tilde{\vee}$, given for an element of $V^* \otimes V^*$ and V^*, as follows:

$$(\sigma^i \otimes \sigma^j)\ \tilde{\vee}\ \sigma^k = \sigma^i \otimes \sigma^j \otimes \sigma^k + \Lambda^{jk}_{lm}\sigma^i \otimes \sigma^l \otimes \sigma^m + \Lambda^{jk}_{lm}\Lambda^{il}_{np}\sigma^n \otimes \sigma^p \otimes \sigma^m \tag{3.7a}$$

$$\sigma^i\ \tilde{\vee}\ (\sigma^j \otimes \sigma^k) = \sigma^i \otimes \sigma^j \otimes \sigma^k + \Lambda^{ij}_{lm}\sigma^l \otimes \sigma^m \otimes \sigma^k + \Lambda^{ij}_{lm}\Lambda^{mk}_{np}\,\sigma^l \otimes \sigma^n \otimes \sigma^p \tag{3.7b}$$

which can be written in the compact form

$$(\sigma^1 \otimes \sigma^2)\ \tilde{\vee}\ \sigma^3 = (I + \Lambda_{23} + \Lambda_{23}\Lambda_{12})\sigma^1 \otimes \sigma^2 \otimes \sigma^3 \tag{3.8a}$$

$$\sigma^1\ \tilde{\vee}\ (\sigma^2 \otimes \sigma^3) = (I + \Lambda_{12} + \Lambda_{12}\Lambda_{23})\sigma^1 \otimes \sigma^2 \otimes \sigma^3 \tag{3.8b}$$

This allows us to calculate $(\sigma^1\ \tilde{\vee}\ \sigma^2)\ \tilde{\vee}\ \sigma^3$ and $\sigma^1\ \tilde{\vee}\ (\sigma^2\ \tilde{\vee}\ \sigma^3)$.

The two previous quantities are equal if and only if Λ satisfies the QYBE:

$$\Lambda_{12}\Lambda_{23}\Lambda_{12} = \Lambda_{23}\Lambda_{12}\Lambda_{23} \tag{3.9}$$

In this case a general q-p-tensor (q-deformed p-tensor) is written as

$$t^{(p),q} = t_{i_1,\ldots,i_p}\,\sigma^{i_1}\ \tilde{\vee}\ \sigma^{i_2}\ \tilde{\vee},\ \ldots,\ \tilde{\vee}\ \sigma^{i_p} \tag{3.10}$$

where $t_{i_1,\ldots,i_p} \in \mathbb{C}$.

Finally one obtains the deformed symmetric associative algebra $S_q(V)$:

$$S_q(V) = \bigoplus_{p=0}^{+\infty} S^{(p),q}(V) \tag{3.11}$$

where $S^{(0),q}(V) = \mathbb{C}$ and $S^{(p),q}(V)$ is the space of q-deformed p-tensors.

3.2. Deformation of the Grassmann Algebra

The exterior product between linear forms is given by

$$\sigma^i \wedge \sigma^j = \sigma^i \otimes \sigma^j - \sigma^j \otimes \sigma^i \tag{3.12}$$

We define the exterior q-product of two linear forms σ^i and σ^j called $\tilde{\wedge}$, by

$$\sigma^i\ \tilde{\wedge}\ \sigma^j = \sigma^i \otimes \sigma^j - \Lambda^{ij}_{kl}\sigma^k \otimes \sigma^l \tag{3.13}$$

A general q-two-form reads

$$w^{(2),q} = w_{ij}\,\sigma^i\ \tilde{\wedge}\ \sigma^j \tag{3.14}$$

where w_{ij} are complex numbers.

To construct a q-form of higher order, we introduce the overlap between the two operations $\otimes$ and $\tilde{\wedge}$, given by El Hassouni *et al.* (1993) for an element of $V^* \otimes V^*$ and V^*, as follows:

$$(\sigma^1 \otimes \sigma^2) \tilde{\wedge} \sigma^3 = (I - \Lambda_{23} + \Lambda_{23}\Lambda_{12})\sigma^1 \otimes \sigma^2 \otimes \sigma^3 \quad (3.15a)$$

$$\sigma^1 \tilde{\wedge} (\sigma^2 \otimes \sigma^3) = (I - \Lambda_{12} + \Lambda_{12}\Lambda_{23})\sigma^1 \otimes \sigma^2 \otimes \sigma^3 \quad (3.15b)$$

This allows us to calculate $(\sigma^1 \tilde{\wedge} \sigma^2) \tilde{\wedge} \sigma^3$ and $\sigma^1 \tilde{\wedge} (\sigma^2 \tilde{\wedge} \sigma^3)$.

In analogy to the symmetric case, the two previous quantities are equal if and only if Λ satisfies the QYBE.

In this case a general q-p-form (q-deformed p-form) is written

$$w^{(p),q} = w_{i_1,\ldots,i_p}\sigma^{i_1} \tilde{\wedge} \sigma^{i_2} \tilde{\wedge}, \ldots, \tilde{\wedge}\sigma^{i_p} \quad (3.16)$$

where $w_{i_1,\ldots,i_p} \in \mathbb{C}$.

Finally one obtain the deformed Grassmann associative algebra $\Omega_q(V)$ as

$$\Omega_q(V) = \bigoplus_{p=0}^{+\infty} \Omega^{(p),q}(V) \quad (3.17)$$

where $\Omega^{(0),q}(V) = \mathbb{C}$ and $\Omega^{(p),q}(V)$ is the space of q-deformed p-forms.

4. STAR PRODUCT AND q-DEFORMATION OF $\Omega(V^*)$ AND OF $S(V^*)$

Let $F(x, y) \in \mathcal{U}(\mathbf{g}) \otimes \mathcal{U}(\mathbf{g})[[h]]$ be a $*$-product on G. This means that $F(x, y)$ satisfies

$$F(x + y, z)F(x, y) = \Phi(x, y, z)F(x, y + z)F(y, z)$$

where $\Phi(x, y, z)$ is ad-invariant. The $R_F(x, y)$ given by

$$R_F(x, y) = F^{-1}(y, x)e^{(ht/2)(x,y)}F(x, y)$$

satisfies equation (2.13).

Now let ρ: $\mathbf{g} \to \mathrm{End}\, V$ be some finite representation of the Lie algebra $\mathbf{g}$; then the matrix

$$B = (\rho \otimes \rho)(R_F) \in \mathrm{End}(V \otimes V)$$

satisfies the QYBE without unitarity condition,

$$B^{12}B^{13}B^{23} = B^{23}B^{13}B^{12} \quad (4.1)$$

which can be written as

$$B^{ij}_{kl}B^{km}_{np}B^{lp}_{qs} = B^{jm}_{kl}B^{il}_{ps}B^{pk}_{ns} \quad (4.2)$$

If we introduce the matrix C defined by $C = \mathcal{P}\ B = \overline{F}^{-1}\mathcal{P}e^{h\overline{t}}\overline{F}$, where $\mathcal{P}$ is the permutation matrix ($\mathcal{P}^{ij}_{kl} = \delta^i_l\delta^j_k$), then equation (4.2) is equivalent to

$$C^{ij}_{kl}C^{lm}_{pq}C^{kp}_{ns} = C^{jm}_{kl}C^{ik}_{np}C^{pl}_{sq} \quad (4.3)$$

which we rewrite in the compact form

$$C_{12}C_{23}C_{12} = C_{23}C_{12}C_{23} \tag{4.4}$$

So C gives rise to a nontrivial representation of the braid groups ρ_n: $B_n \to \mathrm{End}(V^{\otimes n})$. In the case $t = 0$, the corresponding R-matrix given by

$$\overline{S} = (\rho \otimes \rho)(S(x, y))$$

satisfies the triangular QYBE:

$$\overline{S}^{12}\overline{S}^{13}\overline{S}^{23} = \overline{S}^{23}\overline{S}^{13}\overline{S}^{12}, \qquad \overline{S}^{12}\overline{S}^{21} = 1$$

and gives rise to a nontrivial representation of the symmetric group

$$\rho_n: \quad S_n \to \mathrm{End}(V^{\otimes n})$$

Finally, we prove that the matrix $C \in \mathrm{End}(V \otimes V)$ determined by the $*$-product $F(x, y)$ on the (simple) Poisson Lie group G, and depending on the parameter h, leads to a q-deformed exterior product of the algebra $\Omega((V)^*)$ and to a q-deformed symmetric one on $S((V)^*)$.

Let $(\sigma^1, \sigma^2, \ldots, \sigma^n)$ be an arbitrary basis of the finite $\mathbf{g}$-module V, and recall that σ^i can be considered as a linear form on its dual $(V)^*$. We define the products $\tilde{\vee}$, $\tilde{\wedge}$ of σ^i and σ^j, respectively, by

$$\sigma^i \,\tilde{\vee}\, \sigma^j = \sigma^i \otimes \sigma^j + (\mathcal{P}(\rho \otimes \rho)(R_F))^{ij}_{kl}\, \sigma^k \otimes \sigma^l$$

$$\sigma^i \,\tilde{\wedge}\, \sigma^j = \sigma^i \otimes \sigma^j - (\mathcal{P}(\rho \otimes \rho)(R_F))^{ij}_{kl}\, \sigma^k \otimes \sigma^l$$

which can be denoted respectively as

$$\sigma^i \,\tilde{\vee}\, \sigma^j = \sigma^i \otimes \sigma^j + C^{ij}_{kl}\sigma^k \otimes \sigma^l \tag{4.5}$$

$$\sigma^i \,\tilde{\wedge}\, \sigma^j = \sigma^i \otimes \sigma^j - C^{ij}_{kl}\sigma^k \otimes \sigma^l \tag{4.6}$$

where the elements C^{ij}_{kl} are entries of the matrix $C = \mathcal{P}(\rho \otimes \rho)(R_F)$.

From (3.8a) and (4.5) we have

$$\begin{aligned}(\sigma^1 \,\tilde{\vee}\, \sigma^2) \,\tilde{\vee}\, \sigma^3 = (I &+ C_{23} + C_{23}C_{12} + C_{12} + C_{12}C_{23} \\ &+ C_{12}C_{23}C_{12})\sigma^1 \otimes \sigma^2 \otimes \sigma^3\end{aligned} \tag{4.7}$$

By using (3.8b) and (4.5) we obtain

$$\begin{aligned}\sigma^1 \,\tilde{\vee}\, (\sigma^2 \,\tilde{\vee}\, \sigma^3) = (I &+ C_{12} + C_{12}C_{23} + C_{23} + C_{23}C_{12} \\ &+ C_{23}C_{12}C_{23})\sigma^1 \otimes \sigma^2 \otimes \sigma^3\end{aligned} \tag{4.8}$$

From (3.15a) and (4.6) we obtain

$$\begin{aligned}(\sigma^1 \,\tilde{\wedge}\, \sigma^2) \,\tilde{\wedge}\, \sigma^3 = (I &- C_{23} + C_{23}C_{12} - C_{12} + C_{12}C_{23} \\ &- C_{12}C_{23}C_{12})\sigma^1 \otimes \sigma^2 \otimes \sigma^3\end{aligned} \tag{4.9}$$

By using (3.16b) and (4.6) we obtain

$$\sigma^1 \tilde{\wedge} (\sigma^2 \tilde{\wedge} \sigma^3) = (I - C_{12} + C_{12}C_{23} - C_{23} + C_{23}C_{12} \\ - C_{23}C_{12}C_{23})\sigma^1 \otimes \sigma^2 \otimes \sigma^3 \tag{4.10}$$

So, from equation (4.4) we obtain

$$(\sigma^1 \tilde{\vee} \sigma^2) \tilde{\vee} \sigma^3 = \sigma^1 \tilde{\vee} (\sigma^2 \tilde{\vee} \sigma^3)$$
$$(\sigma^1 \tilde{\wedge} \sigma^2) \tilde{\wedge} \sigma^3 = \sigma^1 \tilde{\wedge} (\sigma^2 \tilde{\wedge} \sigma^3)$$

Therefore the products $\tilde{\vee}$ and $\tilde{\wedge}$ defined respectively by (4.5) and (4.6) are associative.

Remark. Note that when the deformation parameter h cancels, $F(x, y)$, $R_F(x, y)$, and $S(x, y)$ are just the identity in $\mathcal{U}(\mathbf{g}) \otimes \mathcal{U}(\mathbf{g})[[h]]$ and

$$B = (\rho \otimes \rho)R_F(x, y) \quad \text{is identity in} \quad \mathrm{End}(V \otimes V)$$

Consequently $C = \mathcal{P}$; then the product $\tilde{\wedge}$ is just the nondeformed exterior product and the product $\tilde{\vee}$ reduces to the classical one.

5. CONCLUSION

A q-deformation of the both the Grassmann and symmetric algebras over a module of a Lie bialgebra can be determined form a star product on the coresponding Poisson–Lie group via the associativity condition, which is equivalent to the quantum Yang–Baxter equation in the two cases.

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REFERENCES

Abe, E. (1980). *Hopf Algebras*, Cambridge University Press, Cambridge.
Akhoumach, K., and Mansour, M. (1996). ICTP preprint IC/IR/96/48, Trieste.
Alvarez-Gaume´, L. (1990). *Nucl. Phys. B Proc. Suppl.* **16**, 571–3.
Baxter, R. J. (1982). *Exactly Solved Models in Statistical Mechanics*, Academic Press, New York.
Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D. (1978). Deformation theory and quantization, *Ann. Phys.* **111**, 61; *Ann. Phys.* **110**, 111.
Bernard, D., and Leclaire, A. (1991). *Commun. Math. Phys.* **142**, 99.
Drinfeld, V. G. (1983). On constant quasiclassical solution of the QYBE, *Math. Dokl.* **28**.
Drinfeld, V. G. (1987). Quantum groups, in Proceedings of the ICM-86 Berkeley, Vol. 1, pp. 789–820.
Drinfeld, V. G. (1990). Quasi-Hopf algebras, *Leningrad Math. J.* **1**, 1419–1457.
El Hassouni, A., Hassouni, Y., and Zakkari, M. (1993). IC/93/156 Miramare-Trieste (June 1993).

Faddeev, L. D. (1984). *Integrable Model in (1+1)-Dimensional Quantum Field Theory*, Elsevier, Amsterdam.

Kac, V. G., and Raina, A. K. (1987). *Bombay Lectures on High Weight Representation of Infinite Dimensional Lie-Algebra*, World Scientific, Singapore.

Mansour, M. (1997). *Int. J. Theoret. Phys.* **36**, 3007–3014.

Mansour, M. (1998). Star-products and quasi-quantum groups, Preprint LPT-ICAC, Rabat, Morocco, and *Int. J. Theor. Phys.* **37**, (1998), 2995.

Moreno, C., and Valero, L. (1992). Star-products and quantization of Poisson–Lie groups, *J. Geophys.* **9**, 369–402.

Takhtajan, L. (1989). *Introduction to Quantum Groups and Integrable Models of Quantum Field Theories*, Springer-Verlag, Berlin.